Remodeling Quantum Measurement

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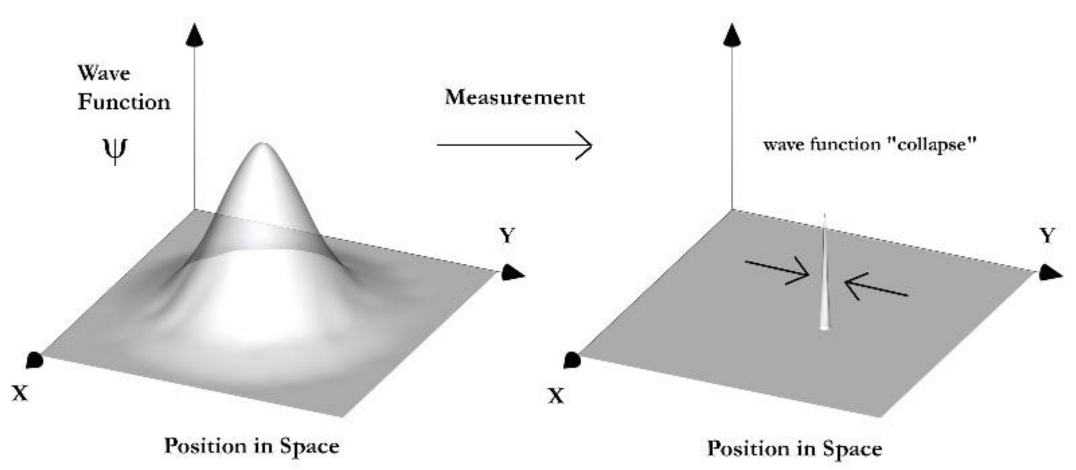
Abstract

Our project is a continuation of Landsman's Flea model as a potential solution to the measurement problem in quantum mechanics. We aim to generalize Landsman's Flea model to different systems, such as periodic potentials and 2-dimensional potentials. In doing so, we also investigate new mathematical tools to understand the effects of the flea.

The Measurement Problem

In quantum mechanics, the properties of a particle (e.g. position, energy) is generally described by a distribution called the wavefunction. A wavefunction, a superposition (linear combination) of many eigenstates, evolves according to Schrödinger's equation, a deterministic process. However, when the particle is measured, the wavefunction "collapses" into one of its eigenstates in a probabilistic way.

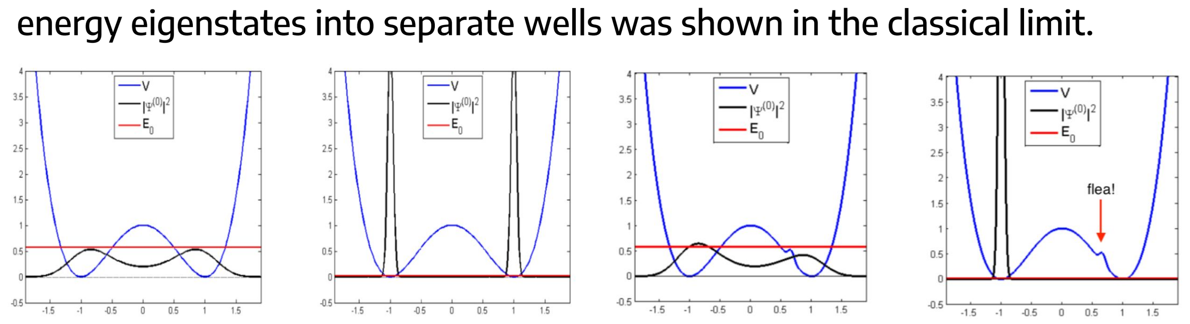


The **Born rule**, a postulate in quantum mechanics, tells us the probability of measuring some eigenstate ψ_k is given by the square modulus of the "amplitude" of ψ_k in the wavefunction:

$$\Psi = \alpha_1\psi_1 + \alpha_2\psi_2 + \cdots + \alpha_n\psi_n \xrightarrow{\text{Measurement}} |\langle\psi_k|\Psi\rangle|^2 = |\alpha_k|^2$$

Landsman's Flea

Our work is largely inspired by 'A Flea on Schrodinger's Cat' - Landsman & Reuvers, in which the measurement process of a discrete 2-state quantum system is modeled by a small perturbation (*flea*) on a double well in the limits of $lim\ t \to \infty$ and $lim\ \hbar \to 0$. Localization of the lowest two energy eigenstates into separate wells was shown in the classical limit.

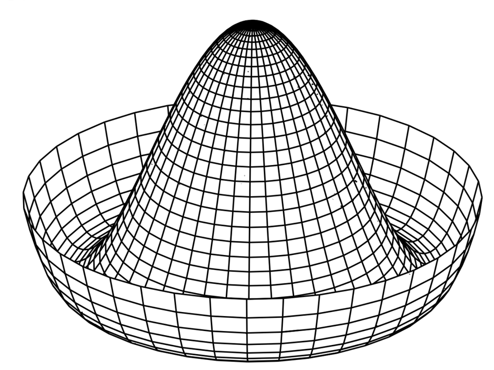


Ground state with no flea, localizes into degenerate classical states (both wells) in the classical limit. *Ground state with flea, localizes into non-degenerate classical states (one well) in the classical limit.*

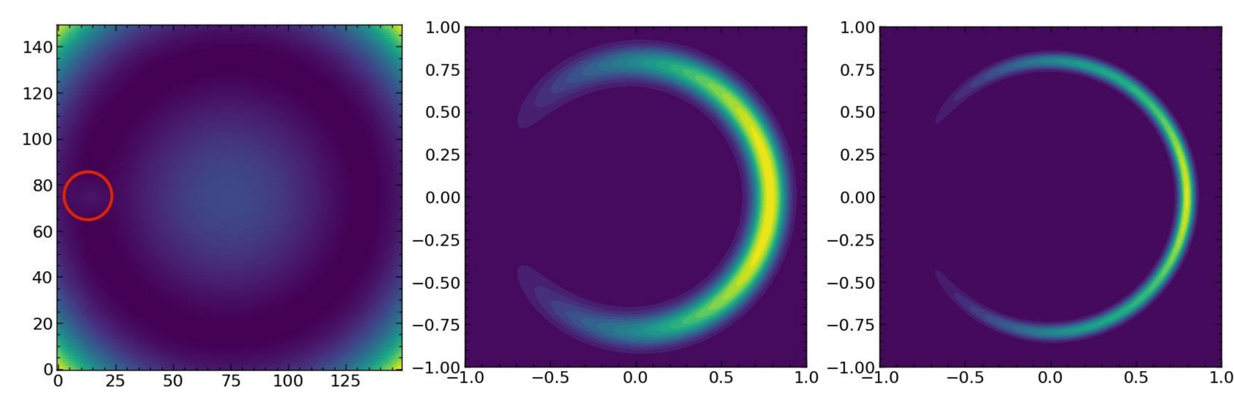
Averaging over possible flea positions, we "re-derive" the Born rule.

2D Well

- We hope to generalize the two-well system results to a 2D system called the "hat potential," which has a continuous circle of lowest potential positions:



- From computer simulations, it looks like adding a flea to this system does give convergence to a point in the $\hbar \to 0$ limit.

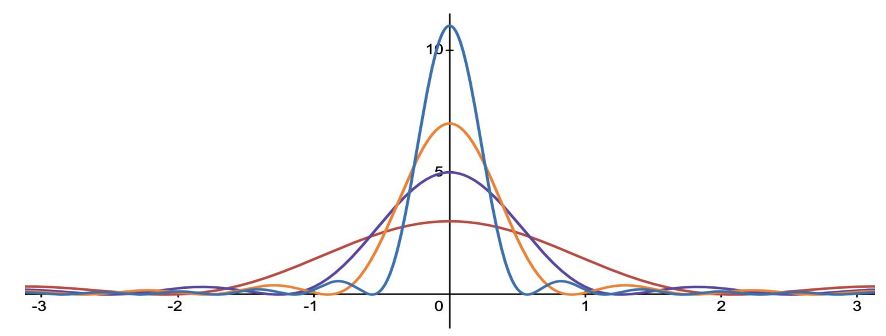


Brighter color corresponds to higher potential energy. Left: hat potential with perturbation circled in red. Middle: resulting probability distribution with ħ = 1. Right: resulting probability distribution with ħ = 0.2.

- The proof of the 2-well case relies on working with approximately localized eigenstates centered in each well.
- For the 2D version, we need to construct a state that's localized at a single point within the bottom circle of the potential.
- This isn't as easy as the 2-well case because we need an infinite sum of eigenstates to get localization at a single point. We hope to approximate the approximately localized state with finite sums:

$$\varphi_{N,\hbar} = C(N,\hbar)\left(\sum_{n=-N}^{N} e^{in\theta}\right)\cdot e^{-\hbar^{-1}(r-1)^2} = C^*(\hbar)\frac{e^{-\hbar^{-1}(r-1)^2}}{\hbar^{1/4}\sqrt{2N+1}}\csc\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}+N\theta\right)$$

- C and C* are just a normalizing coefficients. The angular component is our approximately localized sum and the radial component is just a Gaussian centered at *r = 1*.



The norm squared of the angular component at N=1, N=2, N=3, and N=5.

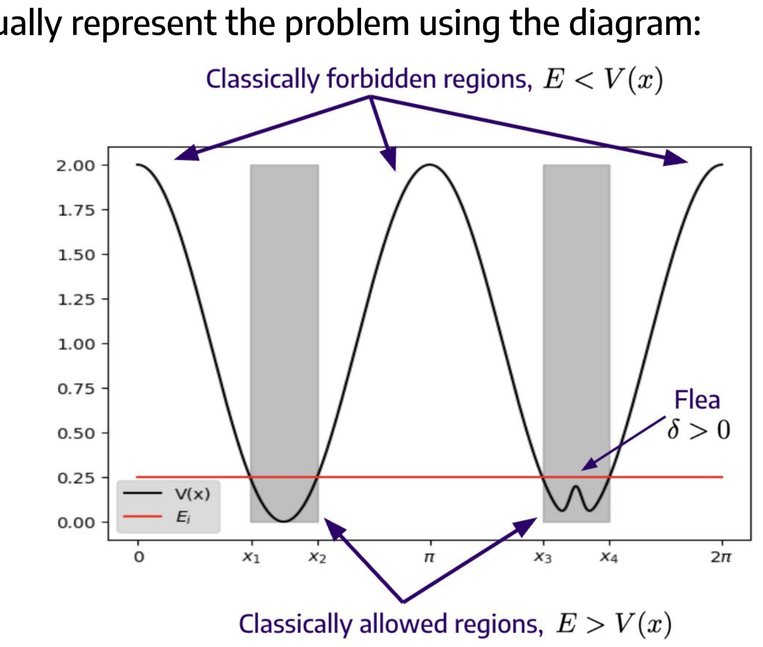
- φ converges pointwise to a delta function as N goes to infinity and $\hbar$ goes to 0. Next we need to show that its Wigner function or Husimi function also has the convergence we want.

Periodic Well WKB

- Attempt to generalize two-well systems to arbitrary numbers of wells, representing the measurement of a discrete n-state system.
- We use following periodic well potential $V(x) = \cos(2x),\ x \in [0, 2\pi]$ and apply the periodic boundary condition: $\psi(0) = \psi(2\pi)$ we analyze the lowest n energy eigenstates each with energy E_i.
- **Goal**: By adding a small, local flea perturbation $\delta > 0$ to one well of the potential, can we prove the wavefunction localizes (to the other wells)? Using previous results, this suffices to prove that Landsman's flea model leads to measurement results following Born's rule.
- Directly analyzing the Hamiltonian of the system proves to be very challenging. Instead, we rely on the WKB approximation.

WKB Approximation

- We visually represent the problem using the diagram:



- After rewriting the Schröginer equation as follows

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi \quad\longrightarrow\quad \frac{d^2\psi}{dx^2} = -\frac{p^2}{\hbar^2}\psi,\quad p(x) = \sqrt{2m[E - V(x)]}$$

and assuming the amplitude of the wavefunction varies slowly, we get

$$\psi(x) \approxeq \begin{cases} \frac{1}{\sqrt{p(x)}}[Ae^{\frac{i}{\hbar}\int_{x_i}^{x} p(x')dx'} + Be^{-\frac{i}{\hbar}\int_{x_i}^{x} p(x')dx'}] & \text{if } E(x) > V(x) \\ \frac{1}{\sqrt{|p(x)|}}[Ce^{-\frac{1}{\hbar}\int_{x_i}^{x} |p(x')|dx'} + De^{\frac{1}{\hbar}\int_{x_i}^{x} |p(x')|dx'}] & \text{if } E(x) < V(x) \end{cases}$$

which is valid away from *turning points* (where $E = V(x)$).

- We want to find the coefficients A_i, B_i or C_i, D_i for each region i. Using $\theta_1 = \frac{1}{\hbar}\int_{x_1}^{x_2} p(x)dx$, $\theta_2 = \theta_1 + \delta$, and $K = \frac{1}{\hbar}\int_{x_2}^{x_3} |p(x)|dx$, asymptotic analysis *connects* coefficients across turning points using

$$T_1 = \begin{pmatrix} e^{-i\theta_1} & 0 \\ 0 & e^{i\theta_1} \end{pmatrix}; T_2 = \begin{pmatrix} e^{-i\theta_2} & 0 \\ 0 & e^{i\theta_2} \end{pmatrix} \quad T_K = \begin{pmatrix} \sqrt{e^{2K}+1}e^{-i\phi} & ie^K \\ -ie^K & \sqrt{e^{2K}+1}e^{i\phi} \end{pmatrix}$$

- Periodicity requires $\begin{pmatrix} A_2 \\ B_2 \end{pmatrix} = T_K T_1 T_K T_2 \begin{pmatrix} A_2 \\ B_2 \end{pmatrix}$, creating the "quantization condition" which describe the energy values of the states and shows energy splitting caused by the flea.

$$\boxed{\theta_1 = \pm\frac{1}{2}\arccos\left(\frac{\cos(\delta) - e^{-2K}}{1 + e^{-2K}}\right) - \frac{\delta}{2} - \phi + \left(n + \frac{1}{2}\right)\pi}$$

- Using this, we get formulas for the wavefunction in each region that can be analyzed/compared in the limit $lim\ \hbar \to 0$.

Husimi Function

- We studied a mathematical tool analogous to that of the Wigner Function for aiding in understanding our classical phase space under specific potential conditions.
- The Husimi (or Q) Function is defined as $Q(q,p) = \frac{1}{\pi}\langle\alpha_{q,p}|\hat{\rho}|\alpha_{q,p}\rangle$ which is the expectation of the coherent state projector. This characterizes a pseudo-probability distribution of position, q, and momentum, p, for our quantum state described by rho.
- To make use of this in our work, we need to understand the dynamics, or time derivative, of this function.
- **Goal**: Our work this quarter was to verify the claim by Drummond that the dynamics of the Q function follow a diffusion evolution represented as a time-symmetric Fokker-Planck equation, and, to further explore under which conditions can we convert this to a forward in time diffusion.

Diffusion Computations

- Drummond's paper claims that diffusion behavior for up to quartic hamiltonians. A significant amount of this quarter was dedicated to verifying the claim for smaller potentials, such as the harmonic oscillator and the free particle.
- We began by understanding the generic dynamic form of Q, namely:

$$\partial_t Q = \frac{1}{\pi}\left[\int(\partial_t\alpha\overline{\psi} + \alpha\partial_t\overline{\psi})\int\overline{\alpha}\psi + \int(\partial_t\overline{\alpha}\psi + \overline{\alpha}\partial_t\psi)\int\alpha\overline{\psi}\right]$$
$$= \frac{1}{\pi}\left[\int(\alpha\partial_t\overline{\psi})\int\overline{\alpha}\psi + \int(\overline{\alpha}\partial_t\psi)\int\alpha\overline{\psi}\right]$$

- And to understand this as diffusion we also characterize how Q evolves in terms of q,p:

$$\partial_\mu Q = \frac{1}{\pi}\left[\int(\partial_\mu\alpha\overline{\psi})\int\overline{\alpha}\psi + \int(\partial_\mu\overline{\alpha}\psi)\int\alpha\overline{\psi}\right]$$

Free Particle Case

- Our analysis makes use of the dynamics of the coherent state itself, where we derive:

$$\partial_x^2\alpha = \left(-\frac{1}{x-q} + \frac{m\omega(x-q)}{\hbar} + \frac{2ip}{\hbar} - \frac{p^2}{m\omega(x-q)}\right)\partial_q\alpha$$

- We prove that by writing these terms into different partials of alpha we can derive that all but the third term cancel and give us symmetry that models Q as evolving with q. It remains an *open* question of how we prove the second term case.

Generalizing / Next Steps

- We can generalize this to the double well potential and the quantum harmonic oscillator, assuming the free particle case to:

$$\hat{H} = -\frac{\hbar^2}{2m}\partial_x^2 + V(x)$$

$$\partial_t Q = \frac{p}{m}\partial_q Q + \frac{1}{\pi}\left[\int\left(\alpha\frac{-1}{i\hbar}\overline{V(x)\psi}\right)\int\overline{\alpha}\psi + \int\left(\overline{\alpha}\frac{1}{i\hbar}V(x)\psi\right)\int\alpha\overline{\psi}\right]$$